

TOPOLOGY 1
MID TERMINAL EXAMINATION

This exam is of 40 marks. Please read all the questions carefully and do not cheat. Please feel free to use whatever theorems you have learned in class after stating them clearly.

1. Show that \mathbb{R}^3 and \mathbb{R} are not homeomorphic. (4)

2. Define $f_n : [0, 1] \rightarrow \mathbb{R}$ by

$$f_n(x) = x^n$$

• Show that $f_n(x)$ converges for every $x \in [0, 1]$ (2)

• Is (f_n) uniformly convergent? (3)

• Is f_n uniformly continuous? (3)

3. Let $f : X \rightarrow Y$ be a continuous function from a topological space X to a compact and Hausdorff topological space Y . Let

$$G_f = \{(x, f(x)) \mid x \in X\}$$

denote the **graph** of f in $X \times Y$. Show that (8)

$$f \text{ is continuous} \Leftrightarrow G_f \text{ is closed in } X \times Y$$

Hint: If G_f is closed and V is a neighbourhood of $f(x_0)$, find a tube around $x_0 \times (Y - V)$ not intersecting G_f .

4. Let \mathbb{Z} denote the set of integers. Define a topology on \mathbb{Z} as follows: $U \subset \mathbb{Z}$ is open if and only if U is \emptyset or is the union of arithmetic progressions:

$$U = \cup_{a,b \mid a \neq 0} S(a, b)$$

where $S(a, b) = \{an + b \mid n \in \mathbb{Z}\}$. Show

• This gives a topology on \mathbb{Z} . (3)

• Finite sets are not open. Hence their complements are not closed. (2)

• The basic open sets $S(a, b)$ are both open and closed. (3)

• $\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} S(p, 0)$. (3)

• There exist infinitely many prime numbers. (3)

5. Let \mathbb{R}^ω denote the countable product of \mathbb{R} . There are three topologies on \mathbb{R}^ω - the box, product and uniform topologies. Under which of these topologies is the function $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ given by

$$f(t) = (t, 1/2t, 1/3t, \dots)$$

continuous? (6)